Applications of Lambert's *W* function to electron device modeling

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IEEE-EDS Orlando Chapter Colloquium, February 2006

Outline:

- Introduction
- Why Lambert's *W* function?
- What is *W* ?
- Some properties of *W*
- Examples of solutions
- Closely related functions
- Explicit non-ideal junction *I-V* models
- Modeling undoped-body MOSFETs
- Other applications
- Summary

Introduction:

- **Many problems in physics are described by implicit lineal-exponential equations.**
- **Closed-form explicit analytic solutions of these transcendental equations, which frequently arise in electronic circuits and device modeling, may be obtained by the use of the Lambert *W* function.**
- **This talk presents a brief review of its mathematical properties and numerical evaluation, and examines some of its applications.**
- **Its usefulness is illustrated in electron device modeling applications, such as non-ideal junctions with parasitic series resistance and shunt conductance, solar cell characteristics, and channel surface potential in bulk and double-gate undoped MOSFETs.**

Motivation:

Advantages of an analytic solution

- Describes general behavior, as opposed to a numerical result that is based on specific initial conditions
- Contributes to unification of different phenomena
- Is exact – although numerical solutions can be as exact as needed
- Can contribute to intuitive understanding
- Has no convergence problems
- Eases understanding behavior as parameters change
- Can be differentiated and integrated

Motivation for *W*:

- **Simple explicit analytic solutions based on *W* eliminate the need for numerical iterative solutions.**
- **Often serve as initial guesses for more complicated iterative, time-dependent, or multi-dimensional calculations.**
- **Make it easier to study perturbations and make the described phenomenon more physically understandable and manageable.**
- **Permit quick evaluation of large number of repetitive cases, especially since many symbolic computation packages (**like *Macsyma®, Mathematica®, Maple®*, etc.) **already contain optimized routines for the Lambert *W* function.**
- **May be readily manipulated since it is possible to explicitly differentiate and integrate *W*.**

Origin of the Lambert *W* function:

It dates back to J. Lambert's work around 1758. Later it was considered by L. Euler in 1779 when he studied Lambert's transcendental equation.

It was denoted "*W*" after the 1959 work of E. M. Wright. Since then it has been used in some applications, but the number has considerably increased over the last few years.

What is *W* ?:

- **The Lambert *W* function is implicitly elementary, that is, it is implicitly defined by an equation containing only elementary functions.**
- ***W* is formally defined for any (complex) z as the (multi-valued, complex) function which is the solution to the simplest transcendental equations that there is, the lineal-exponential equation:**

$$W(z)e^{W(z)} = z$$

What is *W*? :

For real arguments *x*, $W(x)$ is real if $x \geq -e^{-1}$

Definition : $$x = W(x)e^{W(x)}$$

which is to say that if

$$x = ye^{y}$$

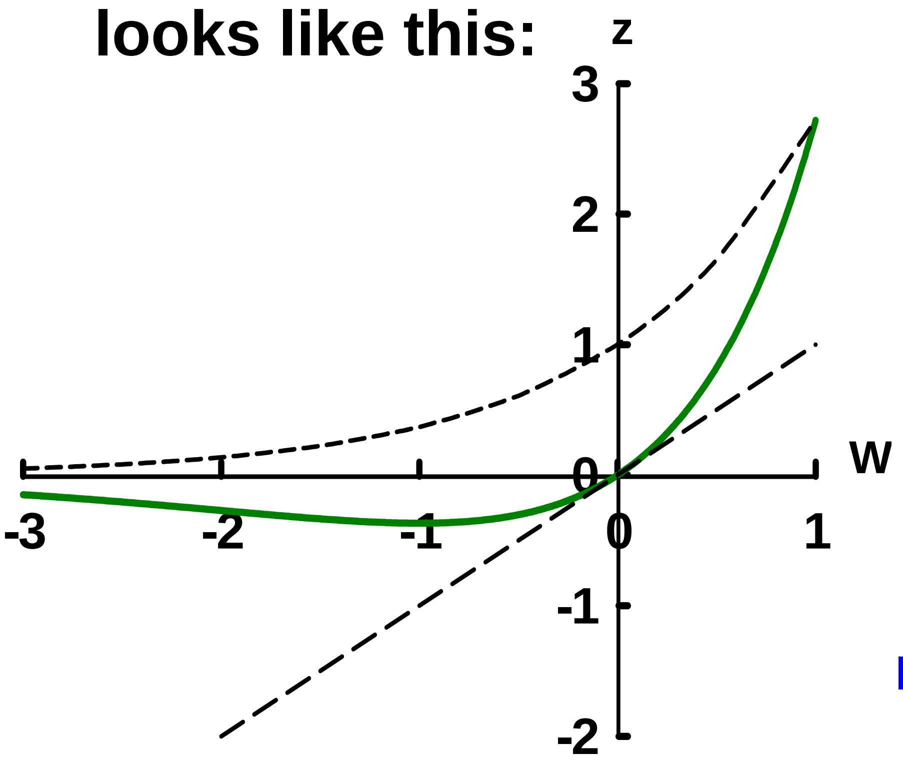
then $$W(x) = y$$

What is *W* ?:

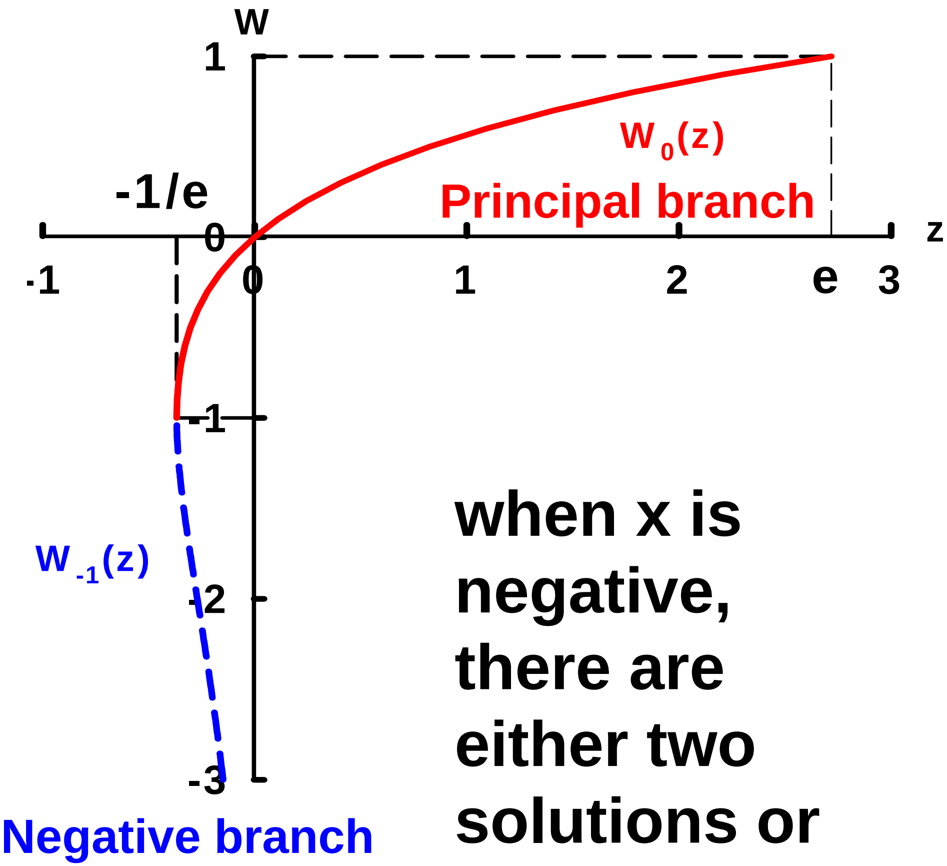
The lineal-exponential equation

$$z = W(z)\,e^{W(z)}$$

looks like this:



By interchanging *z* and *W* axes we get



when x is negative, there are either two solutions or none.

Representations of *W* :

Series about z=0

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n = z - z^2 + \frac{3}{2} z^3 - \frac{8}{3} z^4 + \frac{125}{24} z^5 - \frac{54}{5} z^6 + \frac{16807}{720} z^7 - \dots$$

Asymptote for large z:

$$W_k(z) \approx \ln(z) + 2\pi i k - \ln\left(\ln(z) + 2\pi i k\right)$$

Computation of *W* :

Well-tested, arbitrary-precision numerical software is available for computation of both real branches of the Lambert *W* function in the public domain mathematical library Netlib as Algorithm 743 of the TOMS database (www.netlib.org/toms/743). This single and double precision FORTRAN software computes both real branches of *W* to whatever precision is available on the platform used.

"Algorithm 743: WAPR: a FORTRAN routine for calculating real values of the W-function," by Barry, D. A., Barry, S. J., Culligan-Hensley, P. J., *ACM Trans. Math. Softw.,* vol. 21, pp. 172–181, 1995.

Computation of *W* :

Efficient numerical approximations are contained in the mathematical computer packages: *Maple®* , *Macsyma®* and *Mathematica®*.

***W* is known as the ProdutLog function within *Mathematica®*. The various approximations incorporate iterative schemes to compute *W* with predefined precision, and use piece-wise approximations to generate initial guesses.**

"Global approximations to the principal real-valued branch of the Lambert W-function," by J.P. Body, *Appl. Math. Lett.* 11 (6), pp. 27–31, 1998.

Some examples of solutions using *W*:

Example 1:

Equation: $$x = ye^y$$

Apply *W* to both sides $$W(x) = W\left(ye^y\right)$$

Use definition of *W* $$W\left(ye^y\right) = y$$

Solution: $$y = W(x)$$

Example 2 (first procedure):

Equation: $$x = y\ln(y)$$

Move *y* to the left $$x/y = \ln(y)$$

Exponentiate $$e^{x/y} = y$$

Multiply by *x/y* $$e^{x/y}\, x/y = x$$

Apply *W* $$W\left(e^{x/y}\, x/y\right) = W(x)$$

Use definition $$W\left(e^{x/y}\, x/y\right) = x/y$$

Solution: $$y = \frac{x}{W(x)}$$

Example 2 (second procedure):

Equation: $$x = y\ln(y)$$

Rewrite *y*

$$x = \ln(y)e^{\ln(y)}$$

Apply *W*

$$W(x) = W\left(\ln(y)e^{\ln(y)}\right)$$

Use definition

$$W\left(\ln(y)e^{\ln(y)}\right) = \ln(y)$$

Solution: $$y = e^{W(x)}$$

Example 3 (first procedure):

Equation: $$x = \ln(y)/y$$

Move *y* to the left $$xy = \ln(y)$$

Exponentiate $$e^{xy} = y$$

Divide by -*xy* $$-xye^{-xy} = -x$$

Apply *W* $$W\left(-xye^{-xy}\right) = W\left(-x\right)$$

Use definition $$W\left(-xye^{-xy}\right) = -xy$$

Solution: $$y = -\frac{1}{x}W\left(-x\right)$$

Example 3 (second procedure):

Equation: $$x = \ln(y)/y$$

Rewrite *y* and multiply by -1 $$-x = -\ln(y)e^{-\ln(y)}$$

Apply *W* $$W\left(-x\right) = W\left(-\ln(y)e^{-\ln(y)}\right)$$

Use definition $$W\left(-\ln(y)e^{-\ln(y)}\right) = -\ln(y)$$

Solution: $$y = e^{-W\left(-x\right)}$$

Example 4:

Equation: $$x = \ln(y) + y$$

Exponentiate $$e^x = ye^y$$

Apply *W* $$W\left(e^x\right) = W\left(ye^y\right)$$

Use definition of *W* $$W\left(ye^y\right) = y$$

Solution: $$y = W(e^x)$$

The Wright Function ω:

Defined as the solution to the equation:

$$x = \ln(y) + y$$

Solution: $y = \omega(x)$

Equivalence between Wright function and *W*

$$\omega(x) = W(e^x)$$

"The Wright ω Function." by Corless, R. M. and Jeffrey, D. J. in *Artificial Intelligence, Automated Reasoning, and Symbolic Computation* (Ed. J. Calmet, B. Benhamou, O. Caprotti, L. Henocque and V. Sorge). Berlin: Springer-Verlag, pp. 76-89, 2002.

Example 5 (first procedure):

Equation: $$x = y + e^y$$

Exponentiate $$e^x = e^y e^{e^y}$$

Apply *W* to both sides $$W(e^x) = W\left(e^y e^{e^y}\right)$$

Use definition of *W* $$W\left(e^y e^{e^y}\right) = e^y$$

Solution: $$y = \ln\left(W(e^x)\right)$$

Example 5 (second procedure):

Equation: $$x = y + e^{y}$$

Rewrite $$x - y = e^{y-x} e^{x}$$

Rearrange $$(x-y)e^{(x-y)} = e^{x}$$

Apply *W* $$W\left((x-y)e^{(x-y)}\right) = W\left(e^{x}\right)$$

Use definition $$W\left((x-y)e^{(x-y)}\right) = (x-y)$$

Solution: $$y = x - W\left(e^{x}\right)$$

Identity:

Equating Example 5's two solutions

$$\ln\left(W\left(e^{x}\right)\right) = x - W\left(e^{x}\right)$$

This may be written in terms of the Wright Function:

$$\ln\left(\omega(x)\right) = x - \omega(x)$$

In general:

$$\ln\left(W(x)\right) = \ln(x) - W(x)$$

Example 6:

Equation: $$x = e^{y}/y$$

Rearrange $$-ye^{-y} = -1/x$$

Apply *W* $$W\left(-ye^{-y}\right) = W\left(-1/x\right)$$

Use definition of *W* $$W\left(-ye^{-y}\right) = -y$$

Solution: $$y = -W\left(-1/x\right)$$

The glog Function:

is the solution to the equation: $$x = \frac{e^y}{y}$$

$$y = \text{glog}(x)$$

Equivalence between *glog* and *W*:

$$\text{glog}(x) = -W\left(-1/x\right)$$

$$W(x) = -\text{glog}\left(-1/x\right)$$

"A Generalized Logarithm for Exponential-Linear Equations." Kalman, D., *College Math. J.* 32, pp. 2-14, 2001.

Example 7:

Equation: $$x = y^2 e^y$$

Take square root $$\pm\sqrt{x} = y e^{y/2}$$

Divide by 2 $$\frac{y}{2} e^{y/2} = \pm\frac{\sqrt{x}}{2}$$

Apply *W* $$W\left(\frac{y}{2} e^{y/2}\right) = W\left(\pm\frac{\sqrt{x}}{2}\right)$$

Use definition of *W* $$W\left(\frac{y}{2} e^{y/2}\right) = \frac{y}{2}$$

2 Solutions: $$y = 2W\left(\pm\frac{\sqrt{x}}{2}\right)$$

Example 8:

Equation: $$x = e^{y} / y^{2}$$

Take square root $$\pm\sqrt{x} = e^{y/2} / y$$

Rearrange & divide by 2 $$-\frac{y}{2} e^{-y/2} = \pm\frac{1}{2\sqrt{x}}$$

Apply *W* $$W\left(-\frac{y}{2} e^{-y/2}\right) = W\left(\pm\frac{1}{2\sqrt{x}}\right)$$

Use definition of *W* $$W\left(-\frac{y}{2} e^{-y/2}\right) = -\frac{y}{2}$$

2 Solutions: $$y = -2W\left(\pm\frac{1}{2\sqrt{x}}\right)$$

Example 9:

Equation: $$x = y e^{y^2}$$

Square both sides $$x^2 = y^2 e^{2y^2}$$

Multiply by 2 $$2x^2 = 2y^2 e^{2y^2}$$

Apply *W* $$W\left(2x^2\right) = W\left(2y^2 e^{2y^2}\right)$$

Use definition of *W* $$W\left(2y^2 e^{2y^2}\right) = 2y^2$$

2 Solutions: $$y = \pm\sqrt{\frac{1}{2} W\left(2x^2\right)}$$

Example 10:

Equation: $$y = x^{x^{x^{\cdot^{\cdot}}}}$$

Rewrite $$y = x^y$$

Rearrange $$ye^{-y\ln(x)} = 1$$

Multiply by -ln(*x*) $$-y\ln(x)e^{-y\ln(x)} = -\ln(x)$$

Apply *W* $$W\left(-y\ln(x)e^{-y\ln(x)}\right) = W\left(-\ln(x)\right)$$

Use definition $$W\left(-y\ln(x)e^{-y\ln(x)}\right) = -y\ln(x)$$

Solution: $$y = -\frac{1}{\ln(x)} W\left(-\ln(x)\right)$$

Derivatives of *W* :

$$\frac{\partial W(z)}{\partial z} = \frac{W(z)}{z[W(z)+1]}$$

$$\frac{\partial^2 W(z)}{\partial z^2} = -\frac{W(z)^2[W(z)+2]}{z^2[W(z)+1]^3}$$

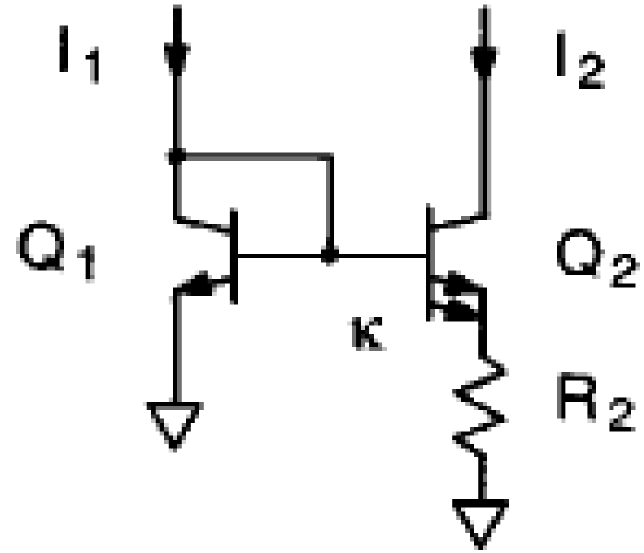
Indefinite integration of *W* :

$$\int W(x)\,dz = \frac{x\left[W(x)^2 + W(x) - 1\right]}{W(x)} + C$$

$$\int xW(x)\,dz = \frac{x^2\left[2W(x)-1\right]\left[2W(x)^2+1\right]}{8W(x)^2} + C$$

APPLICATIONS

Widlar current source:



Equation:

$$I_2 R_2 = v_t \ln\left(\frac{k I_1}{I_2}\right)$$

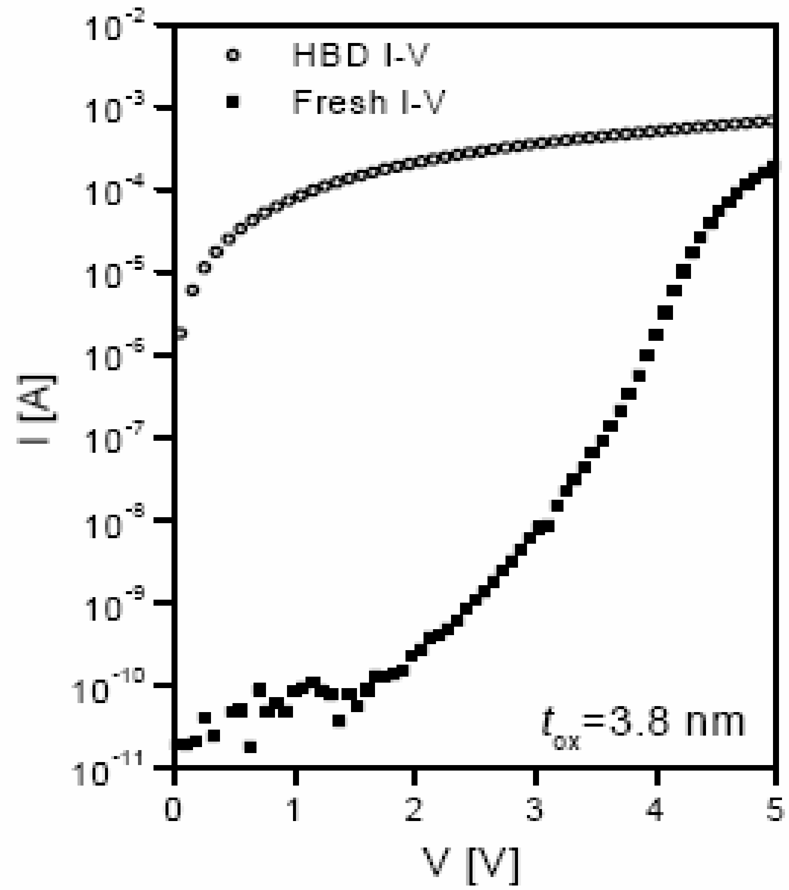
Solution:

$$I_2 = \frac{v_t}{R_2} W\left(\frac{k I_1 R_2}{v_t}\right)$$

From: "Bipolar Transistor Circuit Analysis Using the Lambert W-Function" by Banwell, T.C., *IEEE Trans. Circuits System - I: Fundamental Theory and Applications*, Vol. 47, No. 11, pp. 1621-1633, 2000.

Hard Breakdown in Ultrathin Gate Oxides:

The ever decreasing reliability margins of gate insulators as a consequence of the ongoing miniaturization requires modeling of the post-breakdown current in MOS. A compact representation of the post-breakdown current suitable for circuit simulation environments is given by:



Measured pre- and post- hard breakdown *I-V* characteristics.

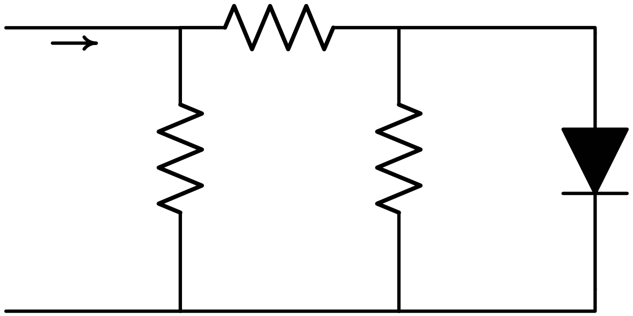
Simulation equation:

$$I = I_0\left\{\exp\left[\alpha\left(V - IR_{BD}\right)\right] - 1\right\}$$

Solution:

$$I = \frac{1}{\alpha R_{BD}} W\left\{\alpha I_0 R_{BD} \exp\left[\alpha\left(V + I_0 R_{BD}\right)\right]\right\} - I_0$$

Non ideal junction :



Equation:

$$I = I_O\left(\exp\left(\frac{V\left(1+R_s\ G_{p2}\right)-I\ R_s}{n\,V_{th}} \right)-1 \right)+\ (V - I\ R_s)\,G_{p1} + V\ G_{p2}\left(1+R_s\ G_{p1}\right)$$

Solutions:

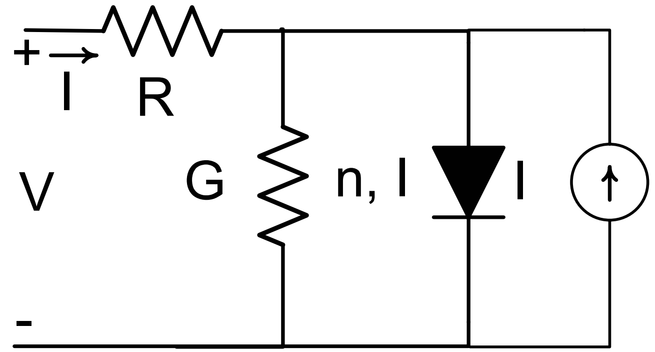
$$I = \frac{nV_T}{R_S} W\left\{ \frac{I_0 R_S}{nV_T\left(1+R_S\ G_{P1}\right)} \exp\left[\frac{\left(V + I_0 R_S\right)}{nV_T\left(1+R_S\ G_{P1}\right)} \right] \right\} + \left(\frac{V\ G_{P1} - I_0}{1+R_S\ G_{P1}} \right) + V\ G_{P2}$$

$$V = -nV_T\ d_2\ W\left\{ \frac{I_0\ R_{12}}{nV_T\ d_2} \exp\left[\frac{\left(I + \frac{I_0}{d_2}\right) R_{12}}{nV_T} \right] \right\} + \ I\ d_2\left[R_S + R_{12}\right] + \ I_0\ R_{12}$$

where

$d_1 \equiv 1/\left(1+R_S\ G_{P1}\right)$ $\quad d_2 \equiv 1/\left(1+R_S\ G_{P2}\right)$ $\quad R_{12} \equiv 1/\left(G_{P1} + G_{P2} + G_{P1}G_{P2}\ R_S\right)$

Solar cell explicit model:



Original implicit equation:

$$I = I_o\left[\exp\left(\frac{V - I R_s}{n V_{th}}\right) - 1\right] + \left(V - I R_s\right) G_p - I_{ph}$$

Explicit solutions:

$$I = \frac{nV_{th}}{R_S} W\left\{\frac{I_0 R_S}{nV_{th}\left(1 + R_S G_P\right)} \exp\left[\frac{V + R_S\left(I_0 + I_{ph}\right)}{nV_{th}\left(1 + R_S G_P\right)}\right]\right\} + \frac{VG_P - \left(I_0 + I_{ph}\right)}{1 + R_S G_P}$$

$$V = -nV_{th} W\left[\frac{I_0}{nV_{th} G_P} \exp\left(\frac{I + I_0 + I_{ph}}{nV_{th} G_P}\right)\right] + I\left(R_S + \frac{1}{G_P}\right) + \frac{I_0 + I_{ph}}{G_P}$$

Solar cell, parameter extraction:
(example of integration usefulness)

Calculate Co-content by integrating explicit *W*-based *I*(*V*) equation:

$$CC(I,V) \equiv \int_0^V \left(I - I_{sc}\right) dV$$

Result:

$$CC(I,V) = C_{\mathrm{V1}}\, V + \mathrm{C}_{\mathrm{I1}} \left(I - I_{sc}\right) + \mathrm{C}_{\mathrm{I1V1}}\, V \left(I - I_{sc}\right) + \mathrm{C}_{\mathrm{V2}}\, V^2 + \mathrm{C}_{\mathrm{I2}} \left(I - I_{sc}\right)^2$$

bidimensional fitting of above equation to experimental data produces the coefficients from which solar cell's parameters can be calculated:

$$R_S = \frac{\sqrt{1 + 16\, \mathrm{C}_{\mathrm{V2}}\, \mathrm{C}_{\mathrm{I2}}} - 1}{4\, \mathrm{C}_{\mathrm{V2}}}$$

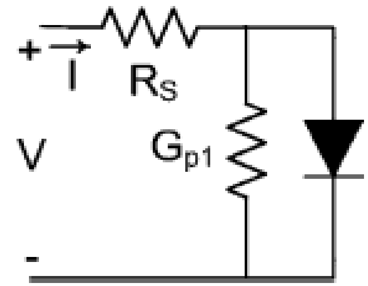
$$G_P = 2\ \mathrm{C}_{\mathrm{V2}}$$

$$I_{ph} = -\frac{\left(1 + \sqrt{1 + 16\, \mathrm{C}_{\mathrm{V2}}\, \mathrm{C}_{\mathrm{I2}}}\right)\left(\mathrm{C}_{\mathrm{V1}} + I_{sc}\right)}{2} - 2\, \mathrm{C}_{\mathrm{I1}}\, \mathrm{C}_{\mathrm{V2}}$$

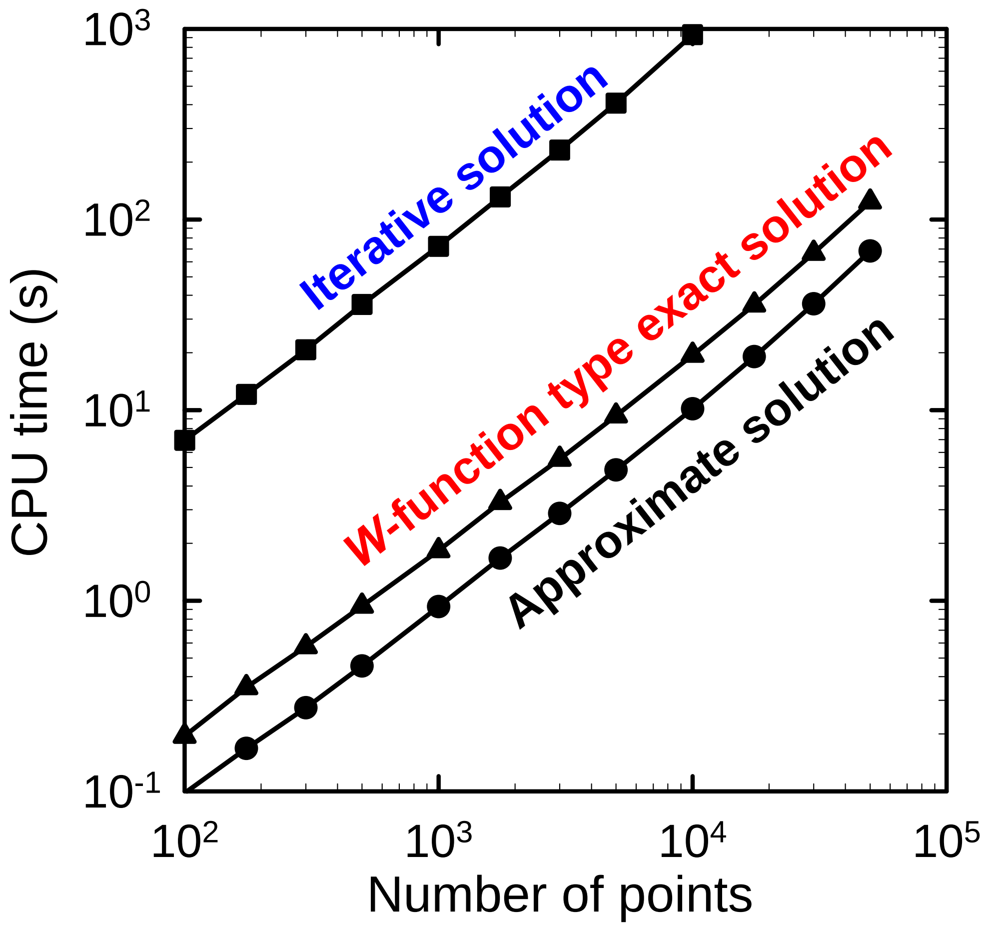
$$n = \frac{C_{\mathrm{V1}} \left(\sqrt{1 + 16\, \mathrm{C}_{\mathrm{V2}}\, \mathrm{C}_{\mathrm{I2}}} - 1\right) + 4\, \mathrm{C}_{\mathrm{I1}}\, \mathrm{C}_{\mathrm{V2}}}{4\, V_{th}\, \mathrm{C}_{\mathrm{V2}}}$$

$$I_o = \frac{I - (V - I\, R_S)\, G_P + I_{ph}}{\exp\left(\dfrac{V - I\, R_S}{n\, V_{th}}\right) - 1}$$

Computation efficiency:



Comparison of non-ideal junction solutions' computation times:



calculated to 10 significant digits

Undoped-body MOSFET Modeling:

- **In conventional doped-body bulk MOSFETs, the surface potential does not have a closed-form exact analytic solution, and must be obtained by numerical iteration or from approximate solutions.**
- **Since most modern Ultra Thin Body MOSFETs will have intrinsic bodies, it is important to model the undoped-body device.**
- **For the case of undoped body MOSFETs, there exists a *W* function-based explicit analytic solution of the surface potential in terms of the terminal voltages.**

Undoped-body bulk MOSFET:

The relationship between gate voltage and channel surface potential for an n-type channel device with undoped body ($n_0 = p_0 = n_i$) is:

$$V_{GF} = \psi_S + \frac{\sqrt{2kT\, n_i\, \varepsilon_s}}{C_o} \sqrt{\mathrm{e}^{-\beta V}\left(\mathrm{e}^{\beta \psi_S} - 1\right)}$$

where $V_{GF} \equiv V_{GS} - V_{FB}$

V_{GS} is the gate voltage

V_{FB} is the flat-band voltage

ψ_S is the surface potential

V is the channel voltage

Undoped-body MOSFET:

Exact analytic solution for an *n*-type channel device with *V*=0 is:

$$\psi_s = V_{GF} + \frac{V_o}{\sqrt{2}} - \frac{2}{\beta} W\left[\frac{1}{4}\beta V_o \sqrt{2}\, e^{\frac{\beta V_{GF}}{2} + \frac{V_o}{\sqrt{2}}}\right]$$

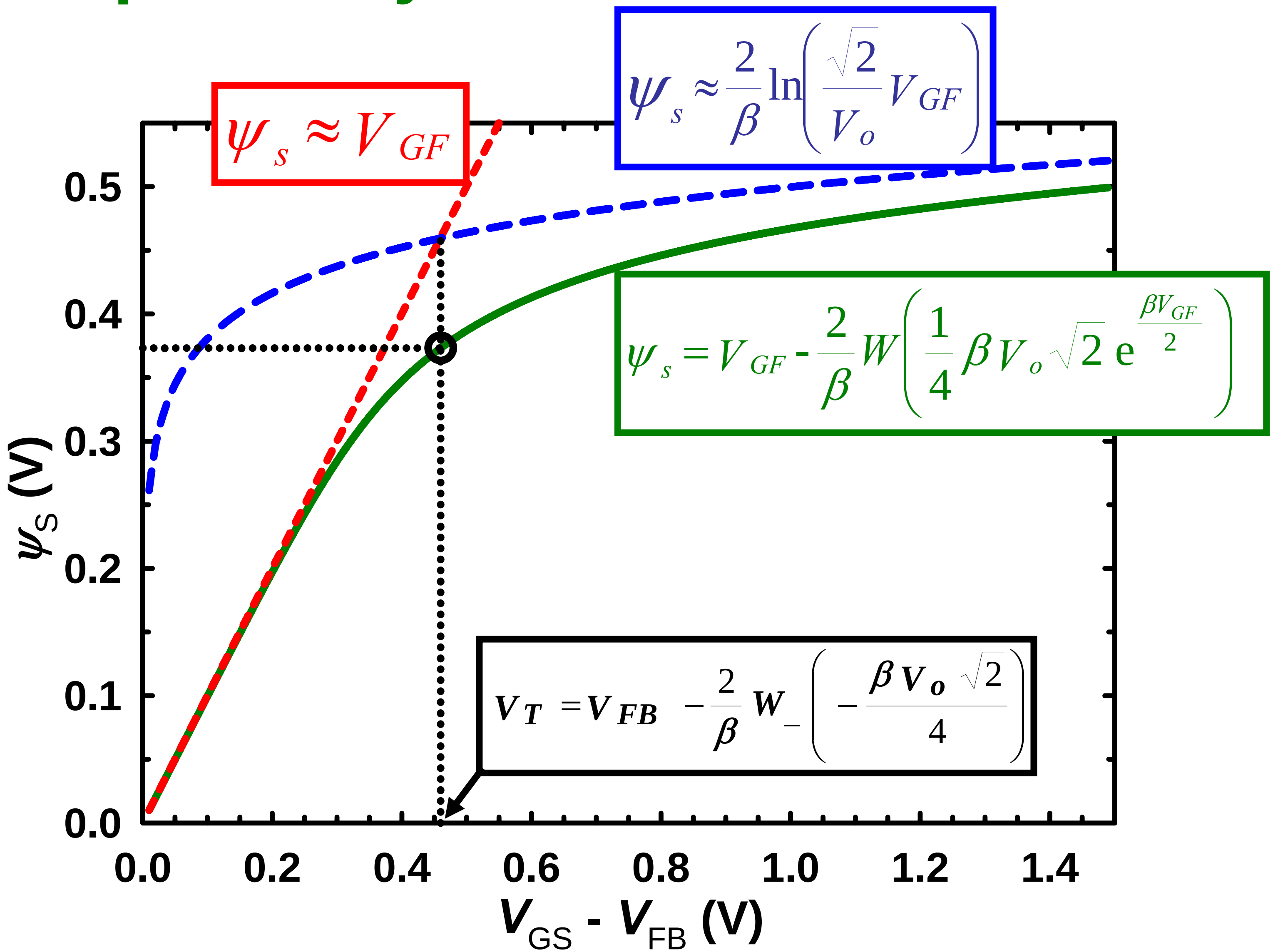
where $V_o = \frac{2\sqrt{kT\varepsilon_s n_i}}{C_o}$ and $\beta = \frac{q}{kT}$

For values of surface potential $\psi_S >> 1/\beta$ we may write an excellent approximation for the *n* -type channel device:

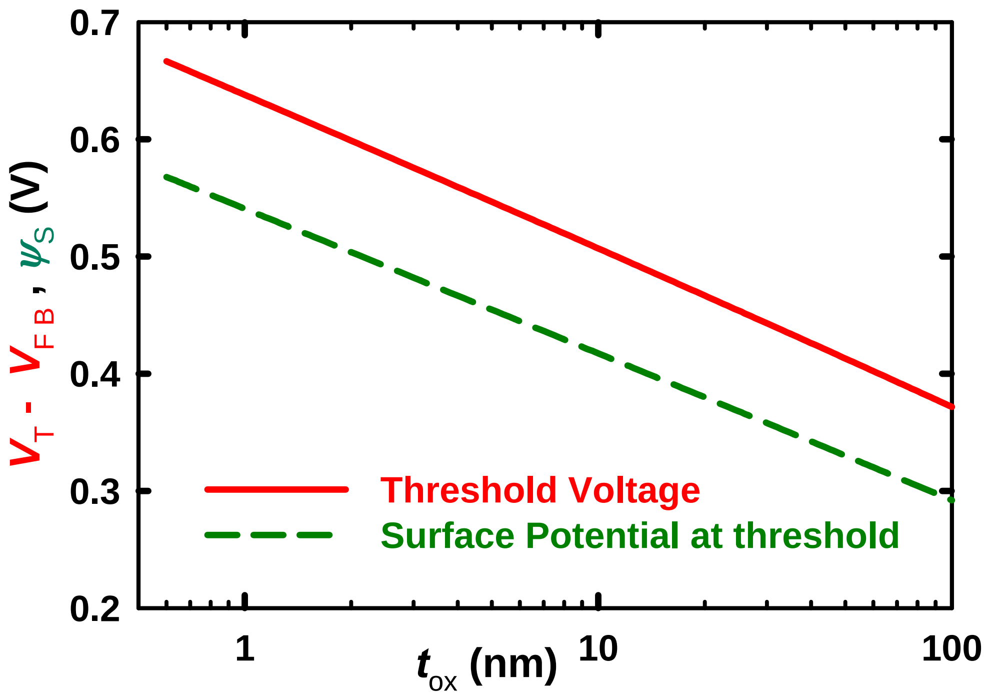
$$\psi_s = V_{GF} - \frac{2}{\beta} W\left[\frac{1}{4}\beta V_o \sqrt{2}\, e^{\frac{\beta V_{GF}}{2}}\right]$$

There are analogous counterparts for the ***p*-type** channel.

Undoped-body MOSFET:



Undoped-body MOSFET - threshold:



V_T decreases as t_{ox} increases
(Opposite trend to that of conventional doped devices)

Led us to predict for the first time that, contrary to conventionally doped devices, the threshold voltage of undoped MOSFETs should decrease as the oxide thickness increases. The same type of behavior was later confirmed for undoped Double-Gate MOSFETs.

Derivatives:

Surface potential and oxide voltage (a), oblique dotted lines correspond to V_{GF} and $V_{GF}/2$.

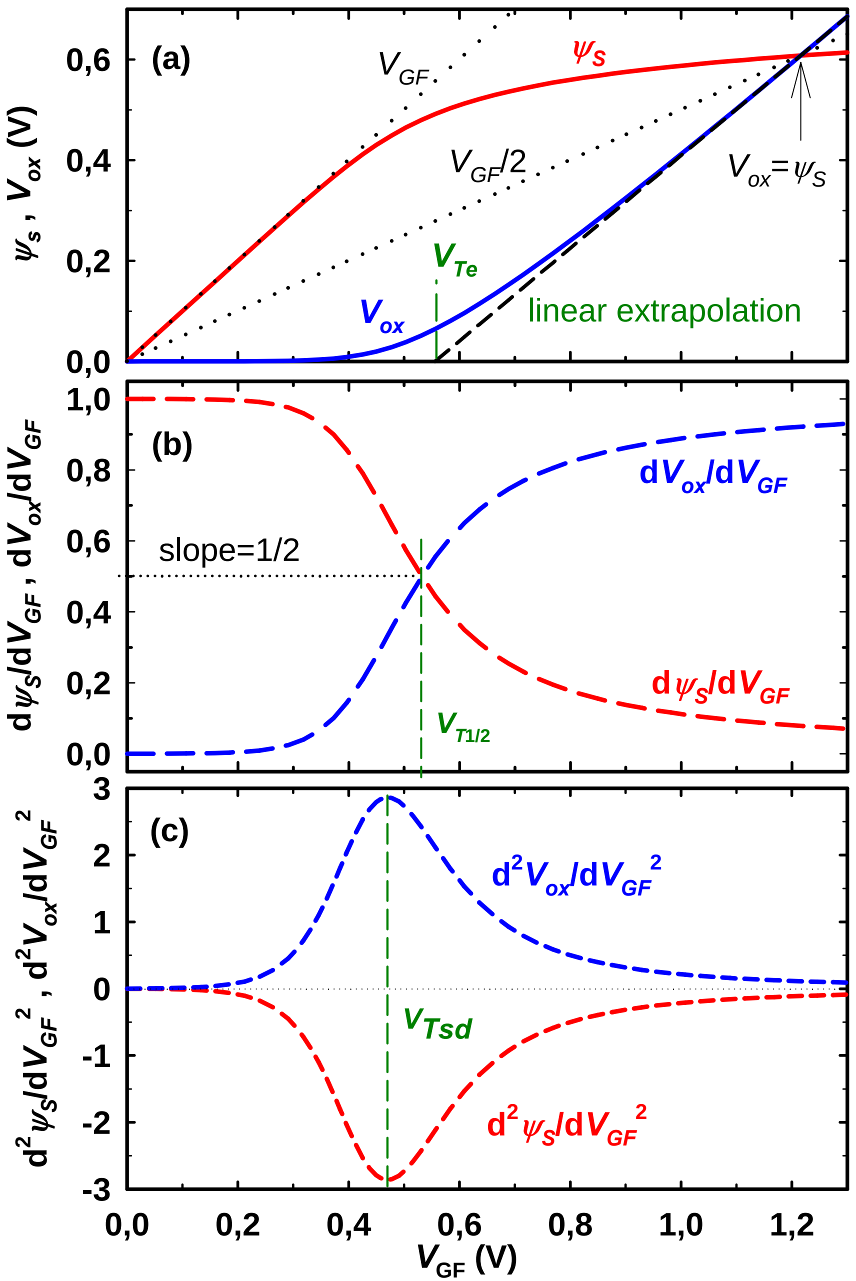
Their first (b) and second (c) derivatives with respect to gate voltage.

Vertical dashed lines indicate threshold voltage definitions:

a: V_{Te} by linear extrapolation from $V_{ox} = \Psi_S$

b: $V_{T1/2}$ at $dV_{ox}/dV_{GF} = d\Psi_S/dV_{GF} = 1/2$

c: V_{Tsd} at the maximum of d^2V_{ox}/dV_{GF}^2



Derivatives - threshold definitions:

* V_{Te} by linear extrapolation from $V_{ox} = \Psi_S$

$$V_{Te} = V_{GFe} - \frac{2}{\beta}\left[1 + W\left(\frac{\beta\, t_{ox}}{\varepsilon_{ox}}\sqrt{\frac{kT\, n_i\, \varepsilon_{si}}{2}\, e^{\beta V_{GFe}}}\right)\right]$$

* $V_{T1/2}$ at $dV_{ox}/dV_{GF} = d\Psi_S/dV_{GF} = 1/2$

$$V_{T1/2} = \frac{kT}{q}\left[2 + \ln(2) + \ln\left(\frac{kT\,\varepsilon_{ox}^2}{n_i \varepsilon_{Si} q^2 t_{ox}^2}\right)\right]$$

* V_{Tsd} at the maximum of d^2V_{ox}/dV_{GF}^2

$$V_{Tsd} = \frac{kT}{q}\left[1 - \ln(2) + \ln\left(\frac{kT\,\varepsilon_{ox}^2}{n_i \varepsilon_{Si} q^2 t_{ox}^2}\right)\right]$$

Undoped-body bulk MOSFET:

The relationship between gate voltage and channel surface potential for all band bending in an undoped body ($n_0 = p_0 = n_i$) is:

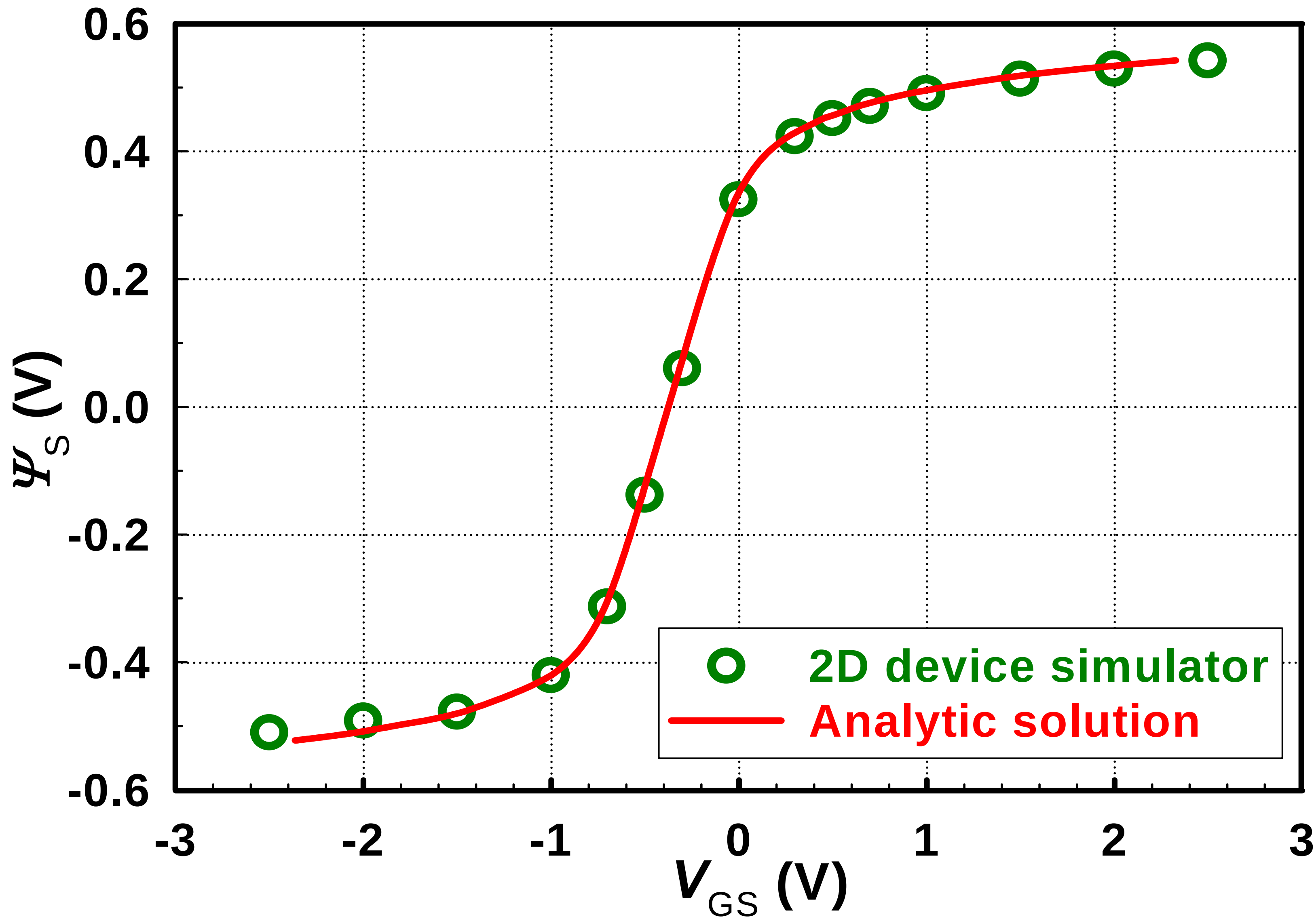
$$V_{GF} = \psi_s + V_0 \sqrt{2} \sinh\left(\frac{\beta \psi_s}{2}\right)$$

where $V_{GF} \equiv V_{GS} - V_{FB}$ and $V_o = \dfrac{2\sqrt{kT\,\varepsilon_s n_i}}{C_o}$

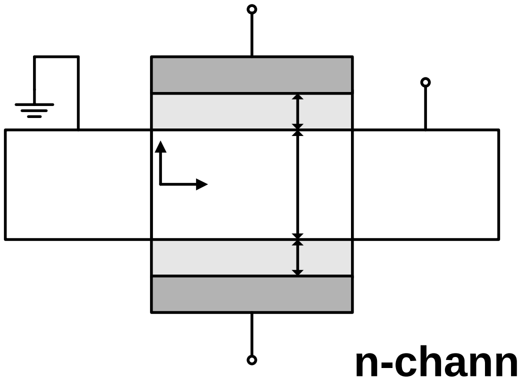
An excellent approximate solution, continuously valid for all band bending, is given ly by:

$$\psi_s = V_{GF} - \frac{2}{\beta} W\left(\frac{1}{4}\beta V_o \sqrt{2}\, e^{\frac{\beta V_{GF}}{2}}\right) + \frac{2}{\beta} W\left(\frac{1}{4}\beta V_o \sqrt{2}\, e^{-\frac{\beta V_{GF}}{2}}\right)$$

Undoped-body bulk MOSFET:



Symmetric DG MOSFET



Both gates have same: work function, oxide thickness and applied bias

Basic equations:

$$V_{GF} = \psi_S + \frac{\sqrt{2kT\, n_i\, \varepsilon_s}}{C_o} \sqrt{\mathrm{e}^{-\beta V}\left(\mathrm{e}^{\beta \psi_S} - \mathrm{e}^{\beta \psi_o}\right)}$$

$$\psi_S = \psi_o - \frac{2}{\beta} \ln\left\{\cos\left[\sqrt{\frac{q^2 n_i}{2kT\, \varepsilon_s}}\, \mathrm{e}^{\frac{\beta(\psi_o - V)}{2}}\, \frac{t_{Si}}{2}\right]\right\}$$

where ψ_O is the center-of-film potential extremum where the electric field = 0

From: "An analytical solution to a double-gate MOSFET with undoped body“ by Y. Taur, in *IEEE Electron Device Lett.*, 21, 245–247, 2000.

Surface Potential in symmetric DG MOSFETs

Combining previous equations:

$$V_{GF} = \psi_S + \frac{\sqrt{2kT\, n_i \varepsilon_s}}{C_o} e^{\frac{\beta(\psi_S - V)}{2}} \sin(\zeta)$$

The solution is given by:

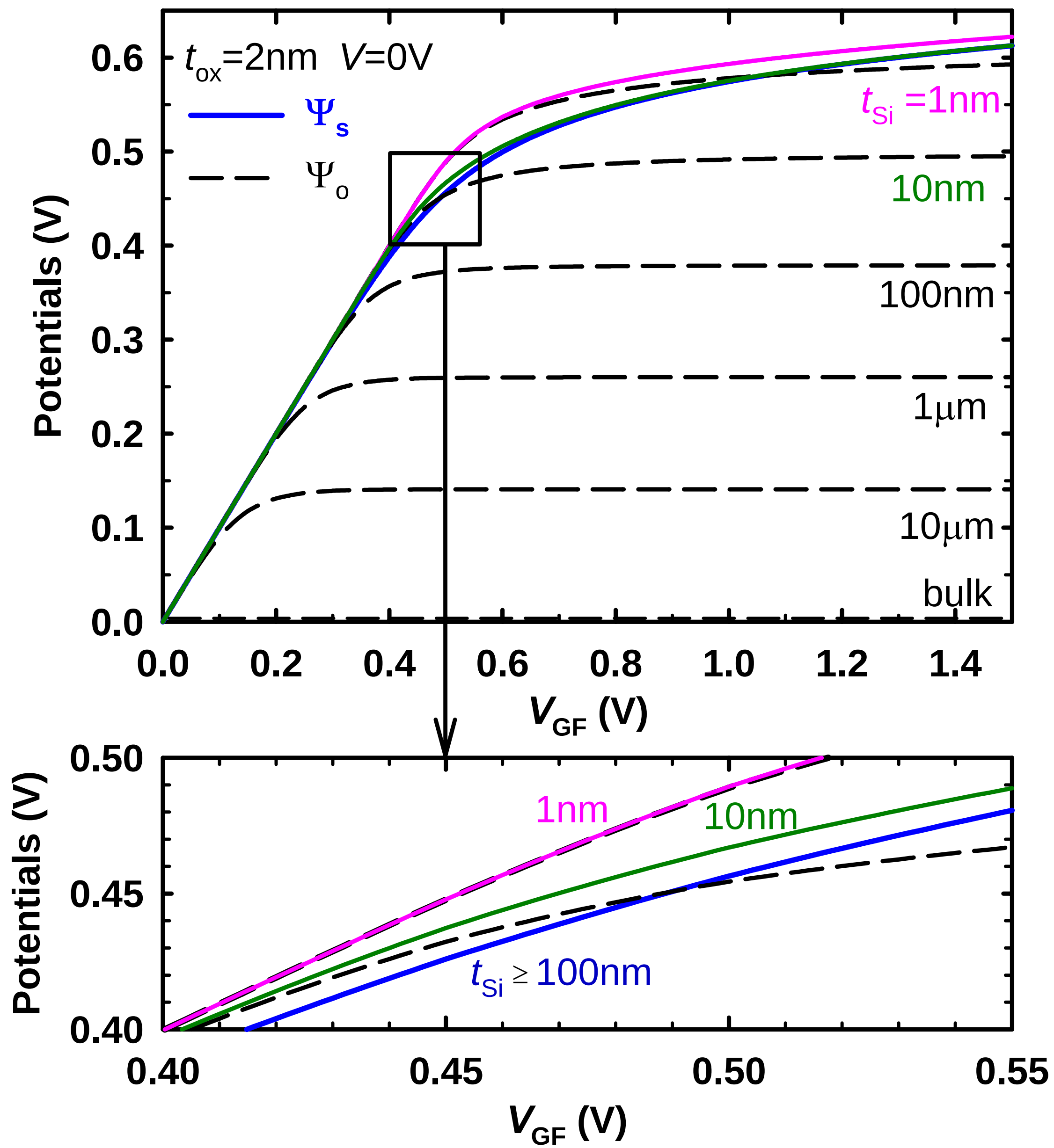
$$\psi_s = V_{GF} - \frac{2}{\beta} W\left[\frac{\beta t_{ox}}{\varepsilon_{ox}} \sqrt{\frac{kTn_i\varepsilon_S}{2}}\, e^{\frac{\beta(V_{GF} - V)}{2}} \sin(\zeta)\right]$$

where $\zeta = \sqrt{\frac{q^2 n_i}{2kT\,\varepsilon_s}}\, e^{\frac{\beta(\psi_o - V)}{2}} \frac{t_{Si}}{2}$

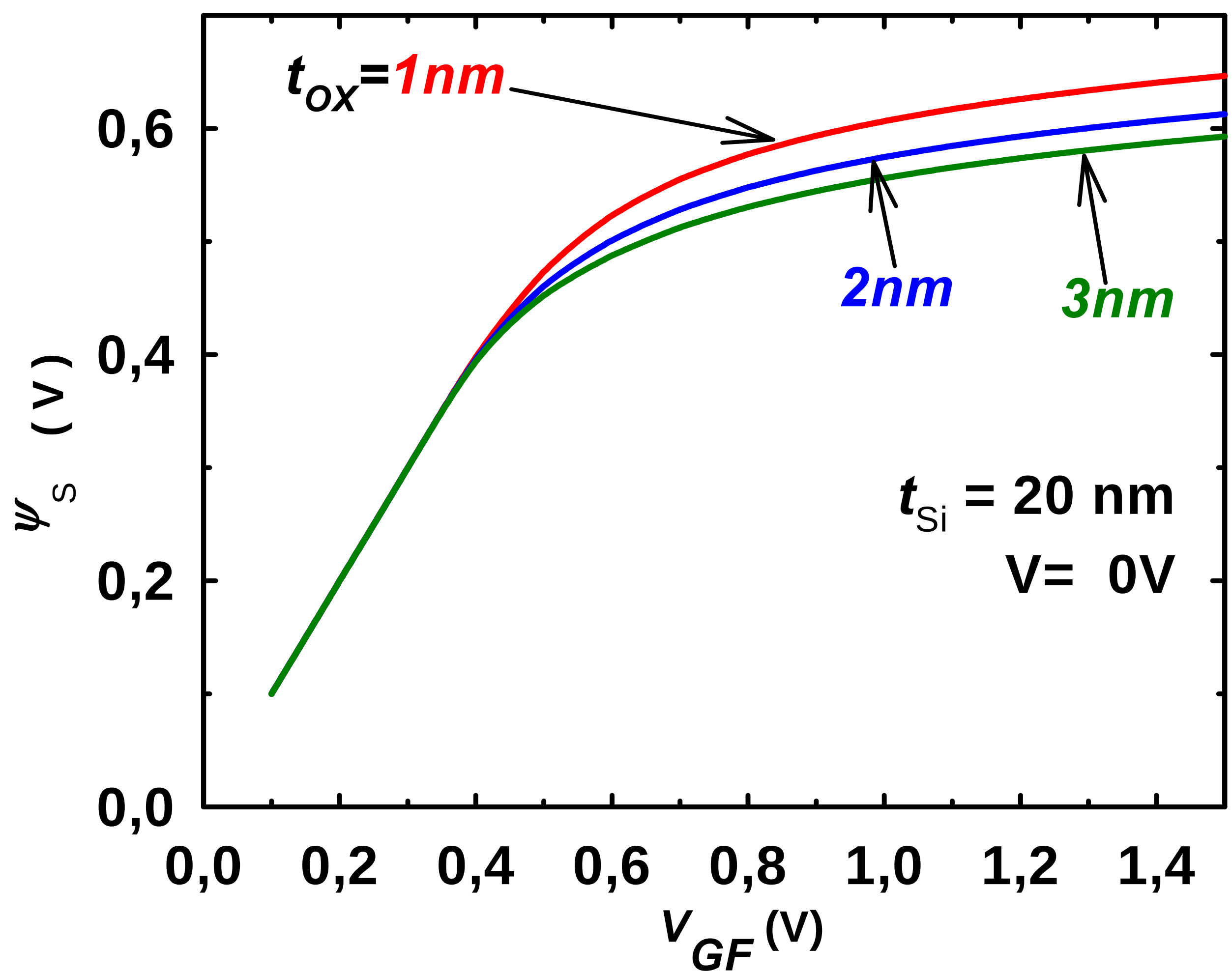
From: "Analytic Solution of the Channel Potential in Undoped Symmetric Dual-Gate MOSFETs", A. Ortiz-Conde, F. J. García Sánchez , and S. Malobabic, IEEE Trans. Electron Device, 52, pp. 1669-1672, July 2005.

Surface & center-of-film potentials:

Gate voltage dependence of the potential extremum, ***Ψ₀*** at the center of the film (dashed lines) and **surface potential**, ***Ψ_S***, (solid lines) for several values of **film body thickness**.



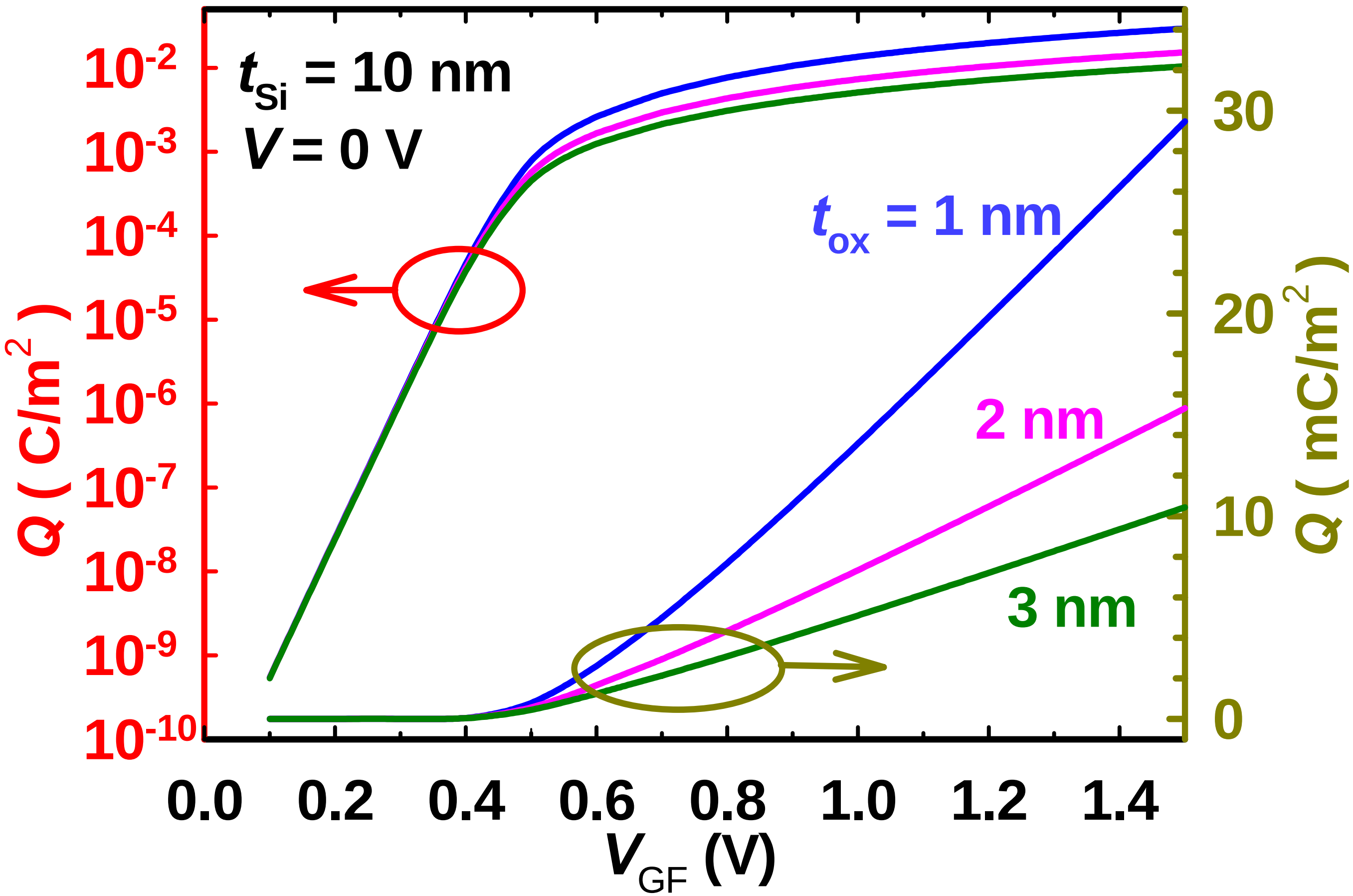
Surface potential:



Surface potential as a function of gate voltage for three values of oxide thickness.

Oxide thickness dependence of DG MOSFET charge :

$$Q = 2\,C_{ox}\,\frac{2}{\beta}\,W\left[\frac{\beta\,t_{ox}}{\varepsilon_{ox}}\sqrt{\frac{kT\,n_i\,\varepsilon_{si}}{2}}\;e^{\beta(V_{GF}-V)}\sin(\zeta)\right]$$

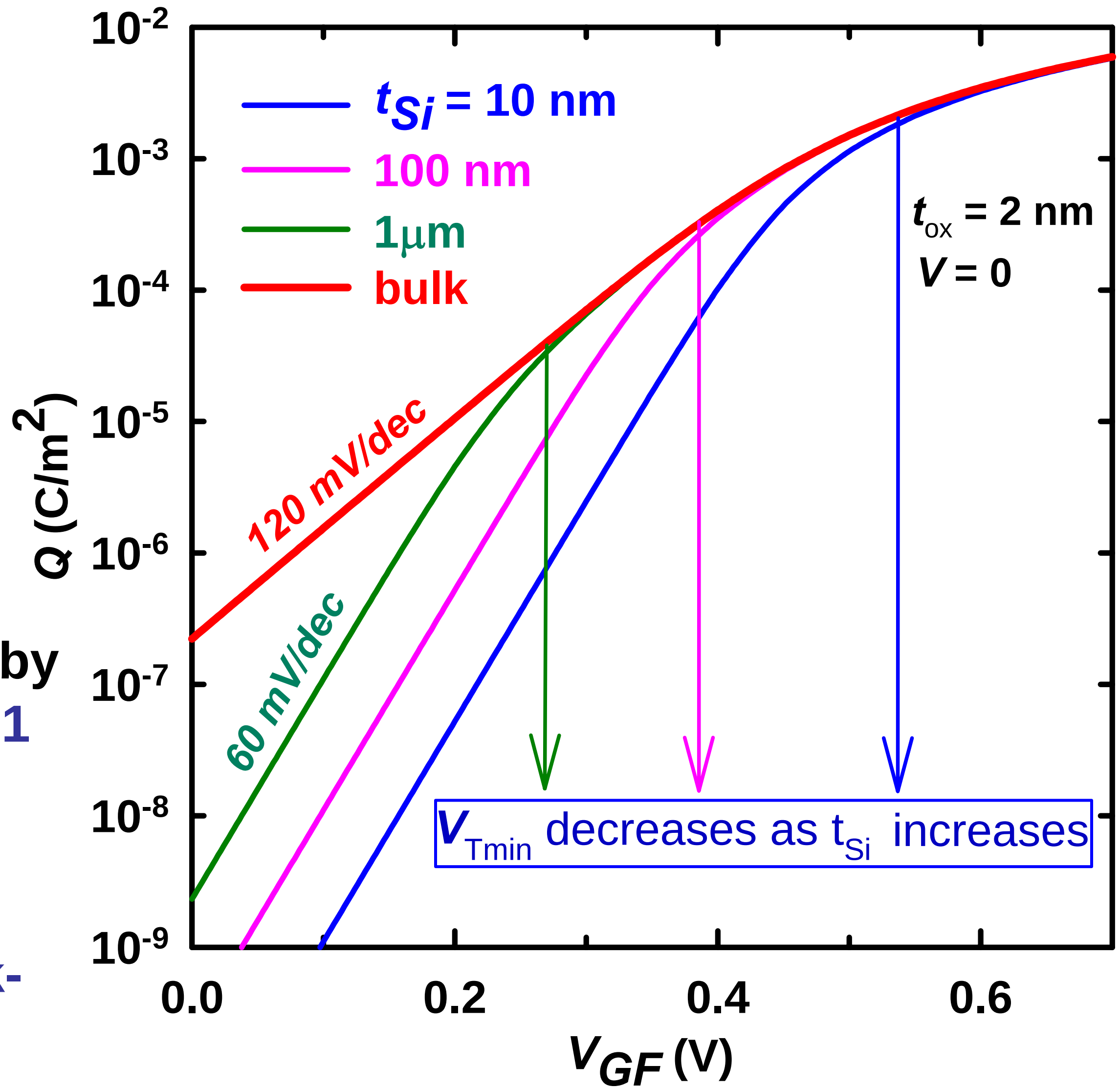


t_{ox} **only affects suprathreshold behavior.**

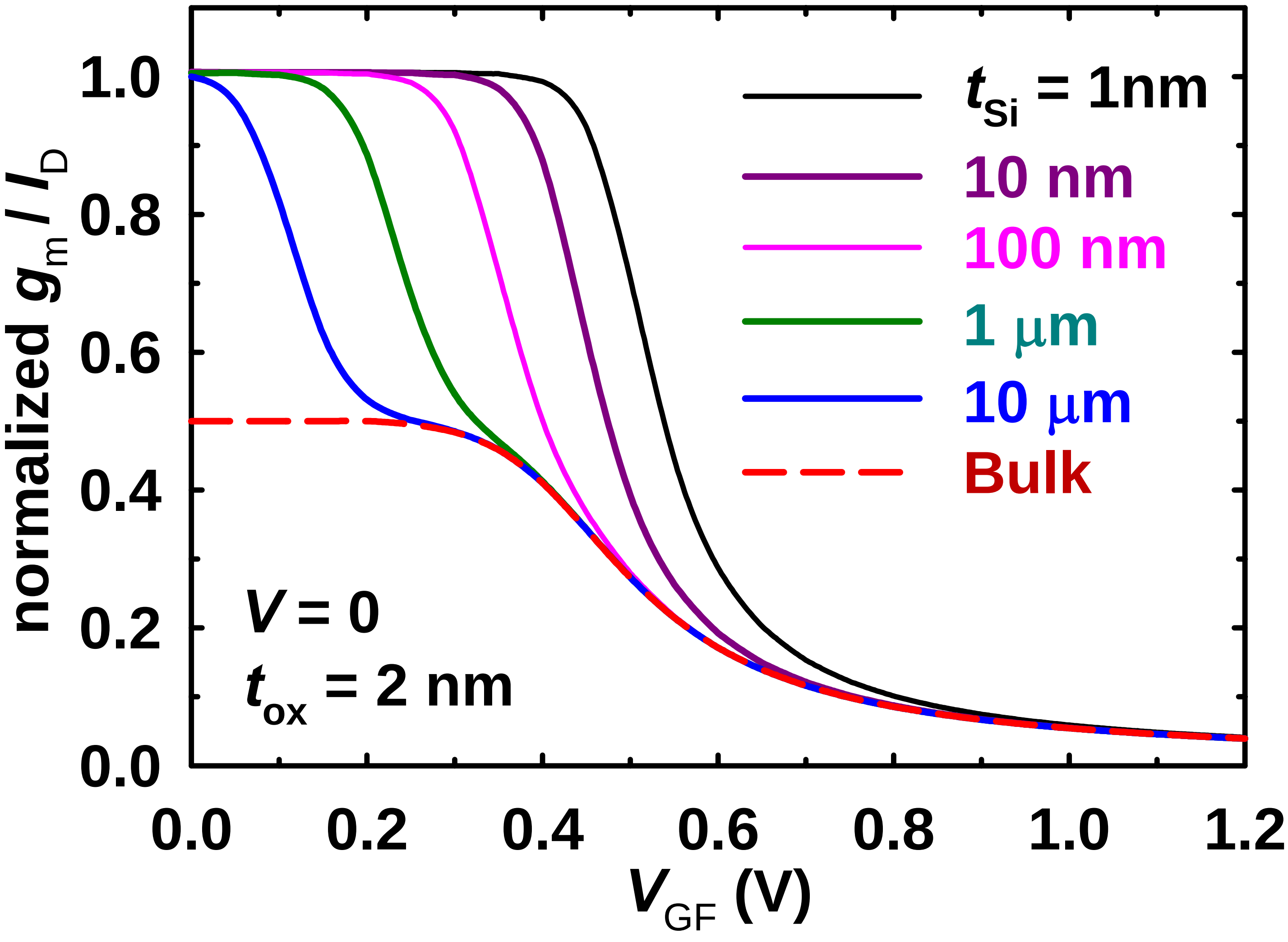
Si film thickness dependence of DG MOSFET charge :

For large t_{Si} , there are two sub-threshold regions (60 and 120 mV/dec). V_{Tmin} is the transition between these two regions.

V_{Tmin} occurs where Q ceases to be governed by volume charge (sin(ζ)<1 DG-like behavior) and starts to be mainly governed by surface charge (sin(ζ)→1 bulk-like behavior).



Transconductance-to-current ratio for various thicknesses



The transconductance to current ratio is equivalent to the inverse of the subthreshold slope factor.

Analytic Surface potential based drain current model:

$$I_D = \mu \frac{W}{L} \Bigg\{ \quad 2C_o \left[V_{GF} \left(\psi_{SL} - \psi_{S0} \right) - \frac{1}{2} \left(\psi_{SL}^2 - \psi_{S0}^2 \right) \right]$$ **(first term)**

$$+ \; 4 \frac{kT}{q} C_o \left(\psi_{SL} - \psi_{S0} \right)$$ **(second term)**

(third term) $$+ \; t_{Si} \, k \, T \, n_i \left[e^{\beta(\psi_{oL} - V_{DS})} - e^{\beta \psi_{o0}} \right] \Bigg\}$$

From: "Rigorous analytic solution for the drain current of undoped symmetric dual-gate MOSFETs“ by A. Ortiz-Conde, F.J. García Sánchez, J. Muci, in *Solid-State Electronics*, 49, 640-647, 2005.

A threshold model:

The intersection of the two current components (I_{Dw} and I_{Ds}) may be understood as the transition threshold from weak to strong conduction.

Below threshold: $\psi_{SL} \approx \psi_{S0} \approx \psi_{oL} \approx \psi_{o0} \approx V_{GF}$ $\quad I_{Dw} = \mu \frac{W}{L} t_{Si}\, q\, n_i \, e^{\beta V_{GF}} \, V_{DS}$

Above threshold: $I_{Ds} = \mu \frac{W}{L} Q_{bulk} \, V_{DS}$ $\quad Q_{bulk} = \frac{4\, C_o}{\beta} W\left(\frac{q\sqrt{2}}{2\, C_o} \sqrt{\frac{\varepsilon_s\, n_i}{kT}} \, e^{\frac{\beta V_{GF}}{2}} \right)$

where Q_{bulk} is the solution for bulk devices.

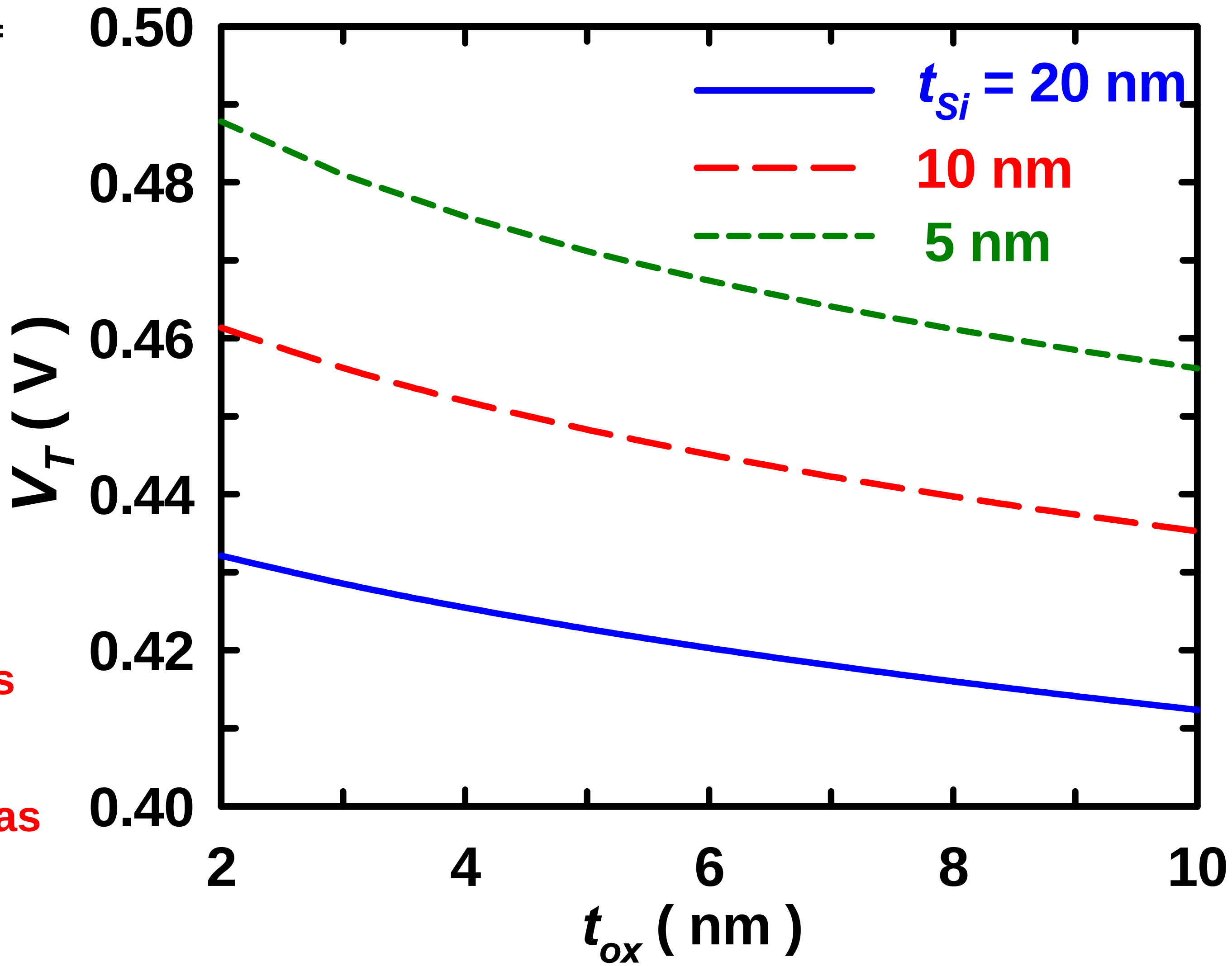
Equating I_{Dw} and I_{Ds} gives:

$$V_T = \frac{kT}{q}\left[\ln\left(\frac{8\, kT\, \varepsilon_s}{q^2 n_i\, t_{Si}^2} \right) - W\left(\frac{4\, \varepsilon_s}{C_o\, t_{Si}} \right) \right]$$

From: "Exact analytical solution of channel surface potential as an explicit function of gate voltage in undoped-body MOSFETs using the Lambert W function and a threshold voltage definition therefrom“ by A. Ortiz-Conde, F.J. García Sánchez, M. Guzmán, in *Solid-State Electronics*, 47, 2067-2074, 2003.

Threshold:

Intersection of the two current components.



V_{Tx} increases as t_{Si} decreases.

V_{Tx} decreases as t_{ox} increases.

Summary:

- **The use of *W* provides simple explicit analytic solutions that eliminate the need for numerical iterative solutions in modeling problems which are described by implicit lineal-exponential equations.**
- **Explicit solutions based on *W* may be readily evaluated and manipulated.**
- **Analytic expressions may be used as initial guesses for more complicated iterative, time-dependent, or multi-dimensional calculations.**
- **The use of *W* makes it easier to study perturbations and dependencies, and turn the described phenomenon more physically understandable and manageable.**
- **Quick evaluation of large number of repetitive cases is made easier for circuit simulation.**
- **It is possible to explicitly differentiate and integrate *W*.**
- **Symbolic computation packages (**like *Macsyma®, Mathematica®, Maple®*, etc.) **already contain optimized routines for *W*.**

END of presentation

Thank you for your attention

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